Christ Church Grammar School

2021
TEST 4

MATHEMATICS SPECIALIST Year 12

Section One:
Calculator-free

Your name SOLUTIONS.

Teacher's name _____

Time and marks available for this section

Reading time for this section:	3 minutes
Working time for this section:	40 minutes
Marks available:	35 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet using a blue/black pen. Do not use erasable or gel pens.

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

See next page

Question 2 **(5 marks)**

Determine $\frac{dy}{dx}$ at the point $\left(\frac{\sqrt{\pi}}{\sqrt{6}}, \frac{\sqrt{\pi}}{\sqrt{3}}\right)$ for the curve defined by the relation

$$\sin(x^2) + \cos(y^2) = \frac{3\sqrt{2}}{\pi} xy.$$

Give your answer in the form $\frac{\pi - a\sqrt{b}}{\sqrt{a}(\pi+\sqrt{b})}$, where $a, b \in Z^+$.

$$\sin x^2 + \cos y^2 = \frac{3\sqrt{2}}{\pi} xy$$

$$2x\cos x^2 - 2y\sin y^2 \frac{dy}{dx} = \frac{3\sqrt{2}}{\pi} y + \frac{3\sqrt{2}}{\pi} x \frac{dy}{dx}$$

✓ differentiates each part correctly

$$2x\cos x^2 - \frac{3\sqrt{2}}{\pi} y = \frac{3\sqrt{2}}{\pi} x \frac{dy}{dx} + 2y\sin y^2 \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x\cos x^2 - \frac{3\sqrt{2}}{\pi} y}{\frac{3\sqrt{2}}{\pi} x + 2y\sin y^2}$$

✓ determines $\frac{dy}{dx}$ based on their derivative

subst in $x = \frac{\sqrt{\pi}}{\sqrt{6}}$, $y = \frac{\sqrt{\pi}}{\sqrt{3}}$

$$\frac{dy}{dx} = \frac{2\frac{\sqrt{\pi}}{\sqrt{6}}\cos\left(\frac{\pi}{6}\right) - \frac{3\sqrt{2}}{\pi}\cdot\frac{\sqrt{\pi}}{\sqrt{3}}}{\frac{3\sqrt{2}}{\pi}\cdot\frac{\sqrt{\pi}}{\sqrt{6}} + 2\cdot\frac{\sqrt{\pi}}{\sqrt{3}}\sin\left(\frac{\pi}{3}\right)}$$

✓ subst. in $\left(\frac{\sqrt{\pi}}{\sqrt{6}}, \frac{\sqrt{\pi}}{\sqrt{3}}\right)$ into their $\frac{dy}{dx}$

$$= \frac{\frac{2\sqrt{\pi}}{\sqrt{6}}\frac{\sqrt{3}}{2} - \frac{\sqrt{3}\sqrt{2}}{\sqrt{\pi}}}{\frac{3}{\sqrt{3}\sqrt{\pi}} + \frac{2\sqrt{\pi}}{\sqrt{3}}\frac{\sqrt{3}}{2}}$$

$$= \frac{\frac{\sqrt{\pi}}{\sqrt{2}} - \frac{\sqrt{3}\sqrt{2}}{\sqrt{\pi}}}{\frac{\sqrt{3}}{\sqrt{\pi}} + \sqrt{\pi}}$$

$$= \frac{\frac{\pi - 2\sqrt{3}}{\sqrt{2}\sqrt{\pi}}}{\frac{\sqrt{3}+\pi}{\sqrt{\pi}}}$$

✓ simplifies to fraction/fraction

$$= \frac{\pi - 2\sqrt{3}}{\sqrt{2}\sqrt{\pi}} \times \frac{\sqrt{\pi}}{\sqrt{3}+\pi} = \frac{\pi - 2\sqrt{3}}{\sqrt{2}(\sqrt{3}+\pi)}$$

$$= \frac{\pi - 2\sqrt{3}}{\sqrt{2}(\pi+\sqrt{3})}$$ ✓ final answer

See next page

Question 1 **(6 marks)**

Determine $\frac{dy}{dx}$ for the following

(a) $y = x^{3x}$. (3 marks)

$\ln y = 3x \ln x$ ✓ Rewrites equations with logs.

$\frac{1}{y}\frac{dy}{dx} = 3\ln x + 3x \cdot \frac{1}{x}$ ✓ differentiates correctly

$\frac{dy}{dx} = y(3\ln x + 3)$

$= 3y(\ln x + 1)$

$= 3x^{3x}(\ln x + 1)$ ✓ correct final answer.

(b) $x = \sin 4\theta$ and $y = -3\cos 2\theta$, leave your answer in terms of θ. (3 marks)

$\frac{dx}{d\theta} = 4\cos 4\theta$ ✓ differentiates correctly

$\frac{dy}{d\theta} = 3\sin 2\theta \cdot 2$

$= 6\sin 2\theta$ ✓ differentiates correctly

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= 6\sin 2\theta \times \frac{1}{4\cos 4\theta}$

$= \frac{3\sin 2\theta}{2\cos 4\theta}$ ✓ Final answer of $\frac{dy}{dx}$ in terms of θ.

See next page

Question 3 **(4 marks)**

Solve the differential equation $\frac{dy}{dx} = \frac{2ye^{2x}}{1+e^{2x}}$ given that $y(0) = \pi$.

$$\frac{dy}{dx} = \frac{2ye^{2x}}{1+e^{2x}}$$

$$\frac{1}{y}\,dy = \frac{2e^{2x}}{1+e^{2x}}\,dx$$ ✓ correctly separates variables

$$\int \frac{1}{y}\,dy = \int \frac{2e^{2x}}{1+e^{2x}}\,dx.$$

If $u = 1+e^{2x}$

$$\frac{du}{dx} = 2e^{2x}$$

$$dx = \frac{du}{2e^{2x}}$$

$$\ln|y| = \ln|1+e^{2x}| + c$$ ✓ integrates correctly with +c

$$y = (1+e^{2x}) \cdot e^{c}$$

Subst $x = 0$, $y = \pi$

$$\pi = (1+e^{0})e^{c}$$

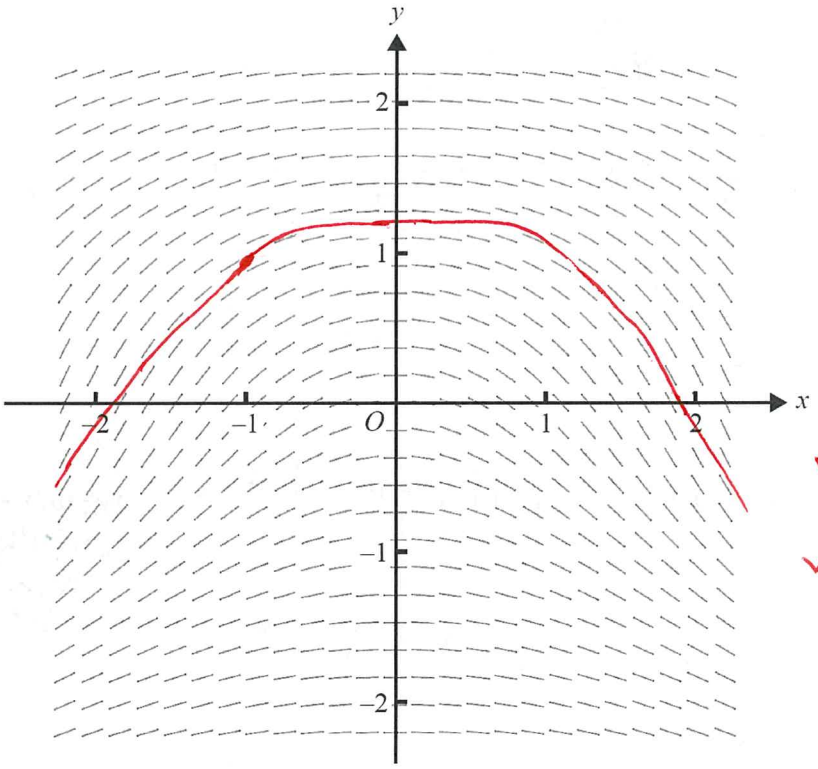
$$\pi = 2e^{c}$$

$$\frac{\pi}{2} = e^{c}$$ ✓ solves their integral to find e^c or other constant.

$$\therefore\ y = \frac{\pi}{2}(1+e^{2x})$$ ✓ writes final equation $y =$.

Question 4 **(6 marks)**

A slope field representing the differential equation $\frac{dy}{dx} = \frac{-x}{1+y^2}$ is shown below.



✓ symmetrical

✓ cuts through x-axis's ≃ ± 1·8

(a) Sketch the solution curve of the differential equation corresponding to the condition $y(-1) = 1$ on the slope field above. (2 marks)

$x = -1$
$y = 1$

(b) Hence, estimate the positive value of x when $y = 0$. Give your answer correct to one decimal place. (1 mark)

$x \simeq 1.8$. ✓

(accept $1.7 \leq x \leq 1.9$)

Question 5 **(8 marks)**

A curve is defined by the equations $x = t^2 + \frac{2}{t}$ and $y = t^2 - \frac{2}{t}$.

Determine the

$x = t^2 + 2t^{-1}$ $y = t^2 - 2t^{-1}$

(a) coordinates of the turning point on the curve. (5 marks)

$\frac{dx}{dt} = 2t - 2t^{-2}$

$= 2t - \frac{2}{t^2}$

$= \frac{2t^3 - 2}{t^2}$ ✓ differentiates correctly (accept either)

$\frac{dy}{dt} = 2t + 2t^{-2}$

$= 2t + \frac{2}{t^2}$

$= \frac{2t^3 + 2}{t^2}$ ✓ differentiates correctly (accept either)

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{2t^3 + 2}{t^2} \times \frac{t^2}{2t^3 - 2}$

$= \frac{2t^3 + 2}{2t^3 - 2}$ ✓ correctly calculates dy/dx as a single fraction

let $\frac{dy}{dx} = 0$ $0 = 2t^3 + 2$ ✓ lets $dy/dx = 0$

$2t^3 = -2$

$t^3 = -1$

$\therefore t = -1$

$x = (-1)^2 + \frac{2}{-1}$

$= -1$

$y = (-1)^2 - \frac{2}{-1}$

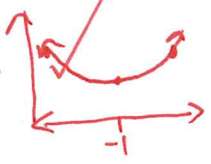
$= 3$

$\therefore (-1, 3)$ ✓ determines co-ordinates

(b) nature of the turning point. (3 marks)

t	-2	-1	-0.5
x	3	-1	$\frac{-15}{4}$
y	5	3	$\frac{17}{4}$
dy/dx	+	0	−

✓ evidence of sign test



evidence of nature

$\therefore$ minimum turning point ✓ states correct nature of turning point.

or correct 2nd derivative ✓
determines concavity ✓
states correct nature ✓

See next page

Question 4 continued

(c) Solve the differential equation $\frac{dy}{dx} = \frac{-x}{1+y^2}$ with the condition $y(-1) = 1$. Express your answer in the form $ay^3 + by + cx^2 + d = 0$, where a, b, c and d are integers. (3 marks)

$$\frac{dy}{dx} = \frac{-x}{1+y^2}$$

$$(1+y^2)\,dy = -x\,dx$$

$$\int (1+y^2)\,dy = \int -x\,dx$$ ✓ correctly sets up separation of variables ∫

or $$\int (1+y^2)\,dy = -\int x\,dx$$

$$y + \frac{y^3}{3} = \frac{-x^2}{2} + c$$

× 6 or (accept either)

$$6y + 2y^3 = -3x^2 + c$$ ✓ correct integration

$$2y^3 + 3x^2 + 6y - c = 0$$

when $x = -1$, $y = 1$

$$2(1)^3 + 3(-1)^2 + 6(1) - c = 0$$

$$2 + 3 + 6 - c = 0$$

$$-c = -11$$

$$c = 11$$

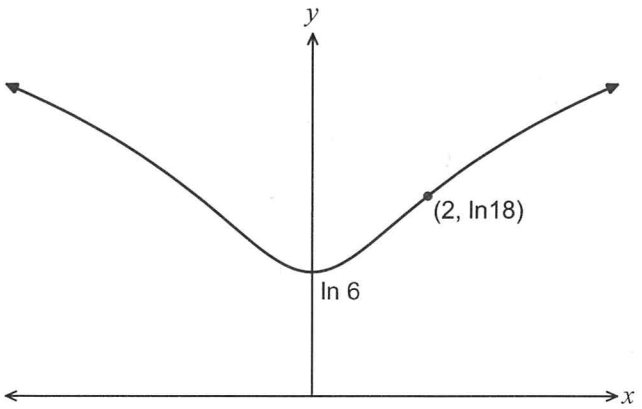
$$\therefore\ 2y^3 + 3x^2 + 6y - 11 = 0.$$ ✓ correct final answer.

See next page

Question 6 **(6 marks)**

The curve below has the gradient function $\frac{dy}{dx} = \frac{2x}{x^2+k}$ for some positive constant k.

Determine the value of k and hence the equation of the curve.



$\frac{dy}{dx} = \frac{2x}{x^2+k}$.

$dy = \frac{2x}{x^2+k}\,dx$

$\int dy = \int \frac{2x}{x^2+k}\,dx$ ✓ sets up separation of variables

$y = \ln|x^2+k| + c$ ✓ integrates correctly

when $x=0$ $y=\ln 6$

$\ln 6 = \ln k + c$ (1)

$x=2$ $y=\ln 18$

$\ln 18 = \ln|4+k| + c$ (2) ✓ substs. in values to set up 2 eqns.

(2) − (1) $\ln 18 - \ln 6 = \ln|4+k| - \ln|k|$

$\ln 3 = \ln\left|\frac{4+k}{k}\right|$

$3 = \frac{4+k}{k} \therefore k = 2$ ✓ solves simultaneously for k

If $k=2$ $c = \ln 6 - \ln 2 = \ln 3$ ✓ solves to determine c

$\therefore y = \ln|x^2+2| + \ln 3$ ✓ states final equation. with their 'c' & 'k' value.

End of questions

Additional working space

Question number: _____

Additional working space

Question number: _____

Christ Church Grammar School

2021
TEST 4

MATHEMATICS SPECIALIST Year 12

Section Two:
Calculator-assumed

Your name SOLUTIONS

Teacher's name _____

Time and marks available for this section

Reading time for this section:	4 minutes
Working time for this section:	40 minutes
Marks available:	39 marks

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, and up to three calculators approved for use in this assessment

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet using a blue/black pen. Do not use erasable or gel pens.

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

See next page

Question 1 **(6 marks)**

Climbing a mountain under ideal conditions, the temperature drops at the rate of $9.8\ ^\circ\text{C}$ per 1000 m. A climber models his ascent on the equation $h = 5t^2 + 150t$, where h is the height in metres and t is the time in hours.

(a) Determine $\frac{dT}{dt}$ in °C per minute. (4 marks)

$\frac{dh}{dt} = 10t + 150$ ✓ calculates $\frac{dh}{dt}$

$\frac{dT}{dh} = \frac{-9.8}{1000}$ ✓ °C/metre states $\frac{dT}{dh}$

$\frac{dT}{dt} = \frac{dT}{dh} \times \frac{dh}{dt}$

$= \frac{-9.8}{1000} \times (10t + 150)$ °C/hour ✓ calculates $\frac{dT}{dt}$ per hour

$= \frac{-9.8}{60000} \times (10t + 150)$ °C/min ✓ calculates $\frac{dT}{dt}$ per min.

$= \frac{-9.8}{6000}(t + 15)$ °C/min

$(-0.00163t - 0.0245)$

(b) Determine the rate, in °C per minute, that the climber experiences the temperature dropping two hours into their ascent. (2 marks)

$t = 2$ hours

$\frac{dT}{dt} = \frac{-9.8}{60000}[10(2) + 150]$ ✓ substs $t = 2$ into $\frac{dT}{dt}$

$= -0.0277666$

drop of $\simeq +0.0278$ °C/min ✓ correct answer with appropriate rounding

Note - Answer only award 2 marks.

See next page

Question 2 **(9 marks)**

The population of an island is currently 154. Its expected growth rate is given by

$$\frac{dP}{dt} = 0.16P\left(1 - \frac{P}{500}\right),$$

where t is in years.

(a) Write P as a function of t. (3 marks)

$\frac{dP}{dt} = 0.16P\left(1 - \frac{1}{500}P\right)$

'k' 'm'

$\frac{dP}{dt} = \frac{0.16P}{500}(500 - P)$ ✓ rewrites to identify 'k' & 'm'

$P = \dfrac{500 \times \frac{154}{500-154}}{\frac{154}{500-154} + e^{-0.00032(500)t}}$ (accept either)

$= \dfrac{500 \times \frac{77}{173}}{\frac{77}{173} + e^{-0.16t}}$ ✓ substs. into formula

$P = \dfrac{222.54}{0.445 + e^{-0.16t}}$ ✓ writes Final equation

or identifies M x y0 ✓

$P = \dfrac{500 \times 154}{154 + (500-154)e^{-\frac{0.16}{500} \cdot 500t}}$ substs. into equation.

$= \dfrac{77\,000}{154 + 346e^{-0.16t}}$ ✓ simplifies + states final equation

or

$P = \dfrac{500}{1 + \left(\frac{346}{154}\right)e^{-0.16t}}$

or $P = \dfrac{500}{1 + 2.25e^{-0.16t}}$ $(2.246753\ldots)$ accept any

(b) Determine the population after 10 years. (2 marks)

$t = 10$ $P = \dfrac{222.54}{0.445 + e^{-0.16(10)}}$ substs into their formula from (a)

$= 343.978$

≈ 344 people ✓ correct number of people

or $P = \dfrac{77000}{154 + 346e^{-0.16(10)}}$

or $P = \dfrac{500}{1 + 0.692e^{-0.16(10)}}$

Note: Correct Answer only award 2 marks.

See next page

Question 2 continued

(c) Determine the time taken for the populations to increase to 480. (2 marks)

$P = 480 \quad \therefore 480 = \dfrac{500}{1 + 2.25e^{-0.16t}}$ ✓ substitutes into their formula from (a)

$\therefore t = 24.9 \text{ years}$ ✓ correct N°. of years. accept 25 years.

Note: Answer only = 2 marks

(d) Determine the limiting population size. (2 marks)

as $t \to \infty \quad e^{-0.16t} \to 0$ ✓ identifies as $t \to \infty$ $e \to 0$

$P \to \dfrac{500}{1+0}$

$= 500$

$\therefore$ limiting population is 500 ✓ states correct value

Note: Answer only award 2 marks.

Question 3 **(6 marks)**

When used in a torch, the lifetime of a single 9V C size battery was observed to be normally distributed with a mean of μ hours and a standard deviation of σ hours.

A student bought 30 boxes of these batteries, with 36 batteries in each box, and calculated the average lifetime for the batteries in each box. The mean of the averages was 30.45 hours and the standard deviation of the averages was 0.38 hours.

(a) Use the information above to determine estimates for μ and σ. (3 marks)

$\mu = 30.45$ h ✓ correctly states mean

correctly identifies ✓ and uses $\sqrt{36}$

$\frac{\sigma}{\sqrt{36}} = 0.38$

$\sigma = 2.28$ h ✓ correct st. dev

The batteries in one of the boxes lasted for a total of 1094 hours.

(b) Use this sample of 36 batteries to construct a 99% confidence interval for the lifetime of this type of battery. (3 marks)

$\bar{x} = \frac{1094}{36}$

$= 30.389$ h ✓ calculates $\bar{x}$

Margin of error $= 2.576 \times 0.38$

$= 0.979$ ✓ calculates margin of error

$\therefore 30.389 \pm 0.979$

$(29.41, 31.37)$ ✓ states CI values

accept either.

or $29.41 \leq \mu \leq 31.37$

or Classpad

C-level 0.99

$\sigma = 2.28$ ✓

$\bar{x} = 30.389$ ✓

$n = 36$

✓ 2 correct values in Classpad

✓ 4 correct values in classpad.

See next page

Question 4 **(9 marks)**

A body moves in a straight line, so that at any time t seconds its displacement, in metres, from a fixed point P on the line is given by

$$x(t) = 12\sin\left(\frac{\pi}{3}t + \frac{\pi}{2}\right), \qquad t \geq 0.$$

The body passes P every T seconds.

(a) Show that body is moving in simple harmonic motion. (3 marks)

$x(t) = 12\sin\left(\frac{\pi}{3}t + \frac{\pi}{2}\right)$

$\dot{x}(t) = 12\cos\left(\frac{\pi}{3}t + \frac{\pi}{2}\right)\cdot\frac{\pi}{3}$

$= 4\pi\cos\left(\frac{\pi}{3}t + \frac{\pi}{2}\right)$ ✓ determines $\dot{x}(t)$

$\ddot{x}(t) = -\frac{4}{3}\pi^2\sin\left(\frac{\pi}{3}t + \frac{\pi}{2}\right)$ ✓ correctly states $\ddot{x}(t)$

$= -\left(\frac{\pi}{3}\right)^2 12\sin\left(\frac{\pi}{3}t + \frac{\pi}{2}\right)$ ✓ shows of multiple of $x(t)$.

$= -n^2x$ (or $-k^2x$)

$\Rightarrow$ SHM.

(b) Determine the value of T. (2 marks)

Period $= \frac{2\pi}{\pi/3}$

$= 2\pi \times \frac{3}{\pi}$

$= 6$ secs. ✓ calculates period

$T = \frac{6}{2} = 3$ Secs ✓ states T

(c) Determine the speed at which the body passes P. (2 marks)

$V^2 = \left(\frac{\pi}{3}\right)(12^2 - 0^2)$ ✓ substs correctly into SHM equation

$= \pm 4\pi$

$\therefore$ Speed $= 4\pi$ m/s $(\simeq 12.6$ m/s$)$ ✓ correctly states speed

(d) Determine the acceleration of the body when $x = 7$ m. (2 marks)

$a = -\left(\frac{\pi}{3}\right)^2(7)$ ✓ Indicates / uses $a = -k^2x$ or $-n^2x$

$= -\frac{7\pi^2}{9}$ m/s^2 ✓ $(\simeq -7.68$ m/s$^2)$ correctly calculates acceleration

Note: correct answer award 2 marks for (b), (c), (d).

See next page

Question 5 **(9 marks)**

The serving sizes of chocolates dispensed by a machine have been observed to have a mean of 220 g and a standard deviation of 3.4 g.

(a) A random sample of 75 serves of chocolates are taken from the machine and the serving size measured in each case. Determine the probability that

(i) the sample mean will be no more than 220.2 g. (3 marks)

Let $\bar{X}$ represent the sample mean.

$\bar{X} \sim N\left(220, \frac{3.4^2}{75}\right)$ ✓ indicates sample mean is normally distributed

$\sigma = \sqrt{\frac{3.4^2}{75}} = 0.3926$ ✓ states correct parameters of normal distribution

$P(\bar{X} < 220.2) = 0.6948$ ✓ states probability

(ii) the total weight of chocolates dispensed will be between 16.482 kg and 16.509 kg. (3 marks)

calculates sample size means ✓ $\frac{16482}{75} = 219.76$ $\frac{16509}{75} = 220.12$

$P(219.76 \leq \bar{X} \leq 220.12) = 0.3496$

writes probability statement ✓ ✓ states probability

(b) After servicing of the machine, an inspector plans to construct a 95% confidence interval for the serving size dispensed by the machine. Determine the sample size they should take so that the width of the interval is no more than 1.5 g, and note any assumptions made. (3 marks)

$z_{0.95} = 1.96$ ✓ indicates correct z-score

$n = \left(\frac{1.96 \times 3.4}{0.75}\right)^2$

$= 78.9$

$\therefore n = 79$ servings ✓ calculates sample size as an integer

assuming standard deviation is still 3.4g and approximating a normal distribution. ✓ writes at least one valid assumption.

End of questions

Additional working space

Question number: _____

Additional working space

Question number: _____